Mean-Variance Reinforcement Learning

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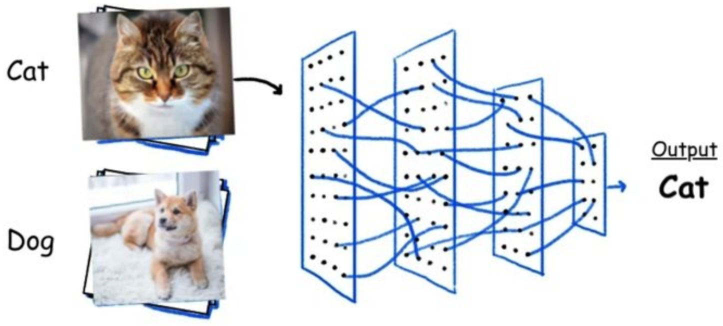
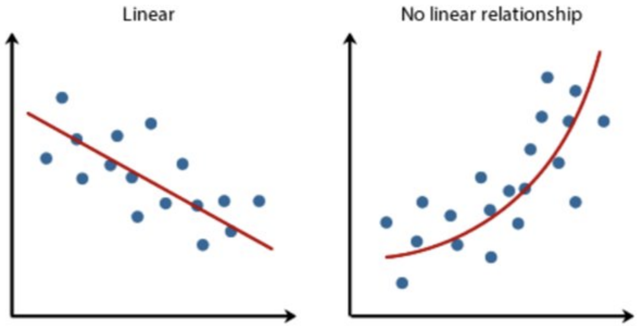
Characteristics of Reinforcement Learning

- Machine Learning, e.g., supervised Learning



Classification



Regression

- $f_\theta(\vec{x}) \quad (\theta = \{\vec{\alpha}, b\})$, e.g, $\vec{\alpha}^\top \vec{x} + b$ Minimize $(\vec{\alpha}^\top \vec{x} + b - y)^2$
- Update parameters: take derivative, and gradient descent (Fit many $(\vec{x}, y)$ to update)
- Classification: apply softmax() function to output, turn value in $[0, 1]$
- Make it Non-linear: stack linear and non-linear layers
  - $f_{\theta_2}(\ \max\{0,\ f_{\theta_1}(\ \max\{0, f_{\theta_0}(\vec{x})\}\ )\ \}\ )$

Characteristics of Reinforcement Learning

What makes reinforcement learning different from other machine learning paradigms?

- No label (no supervisor), only a reward signal (given by the environment)
- Time really matters (sequential, non i.i.d data)
- Agent's actions affect the subsequent data it receives
- Feedback is delayed

Rewards

- Reward R_t is a scalar feedback signal
- Indicates how well agent is doing at step t
- Goal is to maximize cumulative reward $\mathbb{E}[R_1 + \gamma R_2 + \gamma^2 R_3 + ...]$ (discounted)

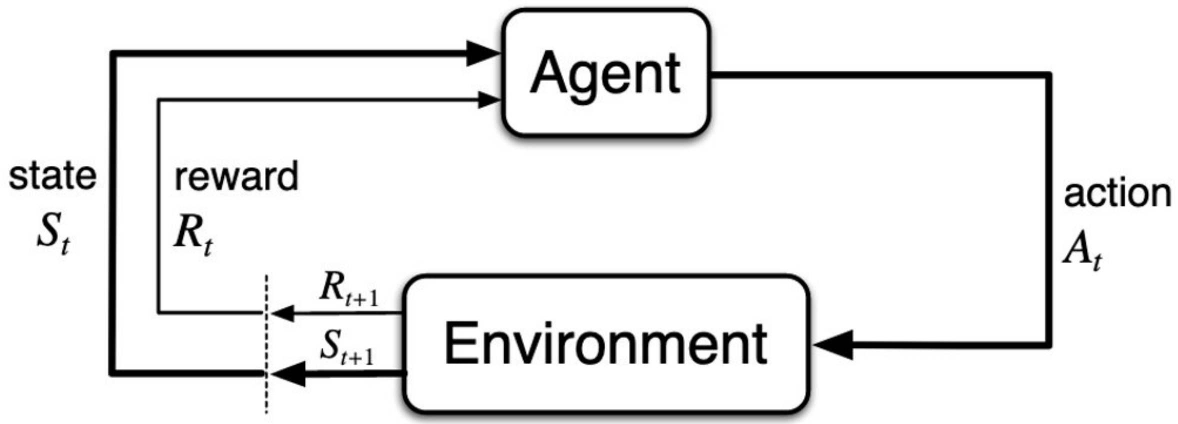
> Reward Hypothesis

All goals can be described by the maximization of expected cummulative reward

- Example: make a robot walk
  - positive reward for forward motion
  - negative reward for falling over

Design reasonable reward for your task

Markov Decision Process (MDP)



A Markov Decision Process is a tuple $\langle \mathcal{S}, \mathcal{A}, \mathcal{P}, \mathcal{R}, \gamma \rangle$

- $\mathcal{S}, \mathcal{A}$ finite set of states, actions
- $\mathcal{P}$ is state transition probability $\mathcal{P}^a_{ss'} = \mathbb{P}[S_{t+1} = s' | S_t = s, A_t = a]$ $(P(s'|s, a))$
- $\mathcal{R}$ is reward function, $\mathcal{R}^a_s = \mathbb{E}[R_{t+1} | S_t = s, A_t = a]$ $(r(s, a))$
- γ is a discount factor $\gamma \in [0, 1]$

Major Components of RL

Policy (usually denote by π):

- A policy is the agent's behavior
- It is a map from state to action $\pi : \mathcal{S} \to \mathcal{A}$
  - Maze. State: location. Action: {Forward, Backward, Left, Right}
- Deterministic policy: $a = \pi(s)$
- Stochastic policy: $\pi(a|s) = \mathbb{P}[A_t = a|S_t = s]$ (good for exploration)
  - Get an action, sample from $\pi(\cdot|s)$

Major Components of RL

Value function

- Value function is a prediction of expected future return
- Evaluate the goodness/badness of states
- Therefore determine which action to choose

$$V^\pi(s) = \mathbb{E}_\pi[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \ldots | S_t = s]$$

$$Q^\pi(s, a) = \mathbb{E}_\pi[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \ldots | S_t = s, A_t = a]$$

$$V^\pi(s) = \mathbb{E}_{a \sim \pi(\cdot|s)}[Q^\pi(s, a)]$$

Bellman Equation

- Bellman Expectation Equation

$$V^{\pi}(s) = \mathbb{E}_{\pi}[R_{t+1} + \gamma V^{\pi}(S_{t+1})|S_t = s]$$

$$Q^{\pi}(s,a) = \mathbb{E}_{\pi}[R_{t+1} + \gamma Q^{\pi}(S_{t+1}, A_{t+1})|S_t = s, A_t = a]$$

Consider a state-action pair (s, a), the reward $r(s, a)$ and all possible next $\{s', a'\}$

$$Q^{\pi}(s,a) = r(s,a) + \gamma \sum_{s'} P(s'|s,a) \sum_{a'} \pi(a'|s') Q^{\pi}(s',a')$$

$$V^{\pi}(s) = \sum_{a} \pi(a|s) \Big(r(s,a) + \gamma \sum_{s'} P(s'|s,a) V^{\pi}(s') \Big)$$

We are evaluating a policy π

Bellman Equation

- Bellman Optimality Equation

The optimal value function is the maximum value over all policies

$$V^*(s) = \max_\pi V^\pi(s) \quad Q^*(s,a) = \max_\pi Q^\pi(s,a)$$

$$V^*(s) = \max_a Q^*(s,a)$$

- Optimal Policy: every state achieves the optimal value (cannot do even better)

$$\pi^*(s) = \arg\max_a Q^*(s,a)$$

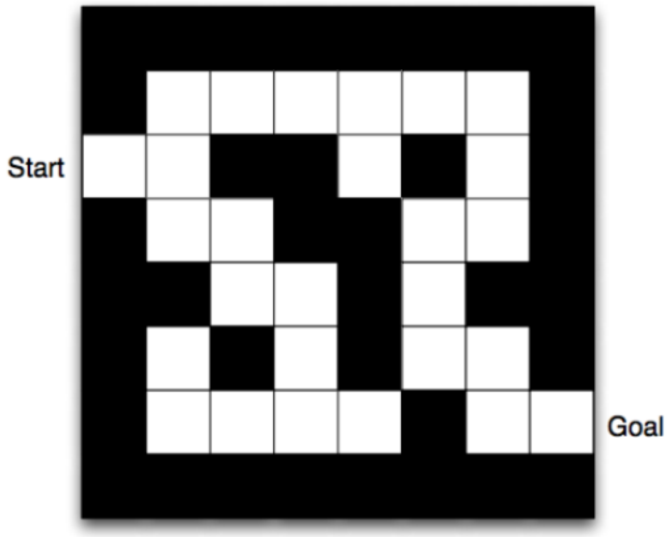
$$Q^*(s,a) = r(s,a) + \gamma \sum_{s' \in \mathcal{S}} P(s'|s,a) \max_{a' \in \mathcal{A}} Q^*(s',a')$$

$$V^*(s,a) = \max_a \left[r(s,a) + \gamma \sum_{s'} P(s'|s,a) V^*(s') \right]$$

$\pi^*(a^*|s) = 1.0$

Find Optimal Policy: value based method

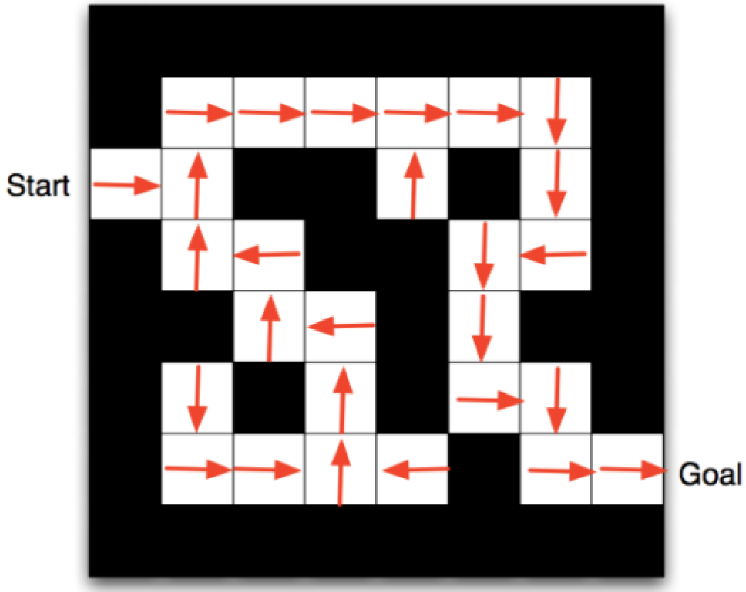
- There is always a deterministic optimal policy for any MDP
- If we know $Q^*(s, a)$, we immediately have the optimal policy

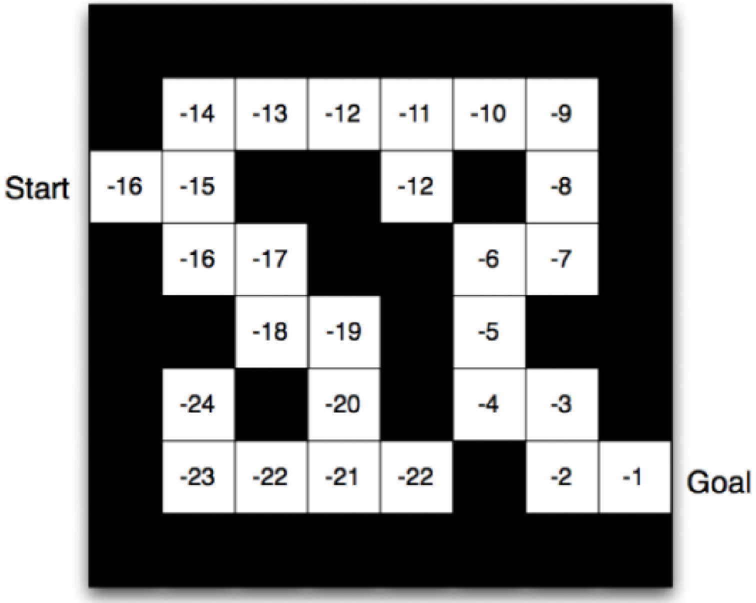


Rewards: -1 per step. Undiscounted ($\gamma = 1$).

State: current location, Actions: four directions

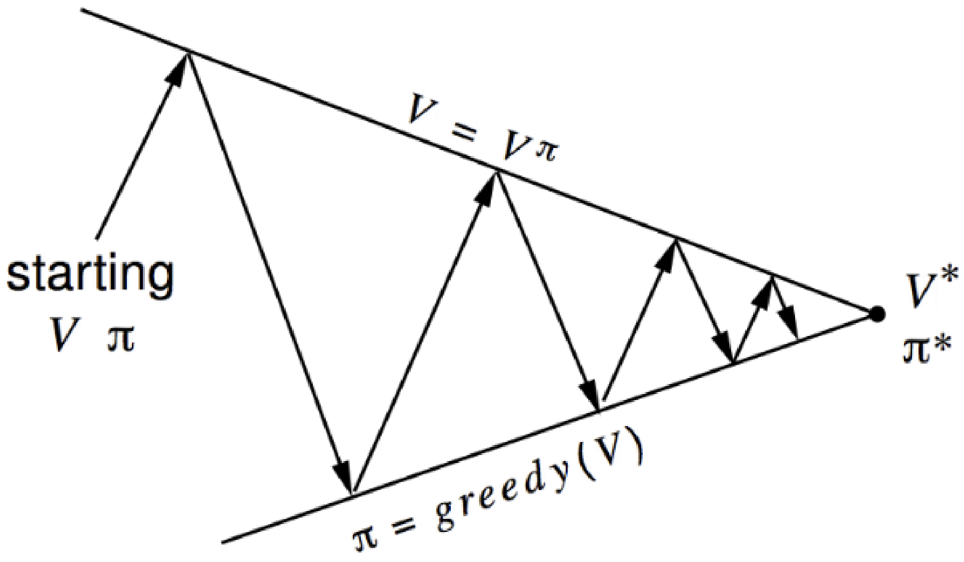
Find Optimal Policy: value based method





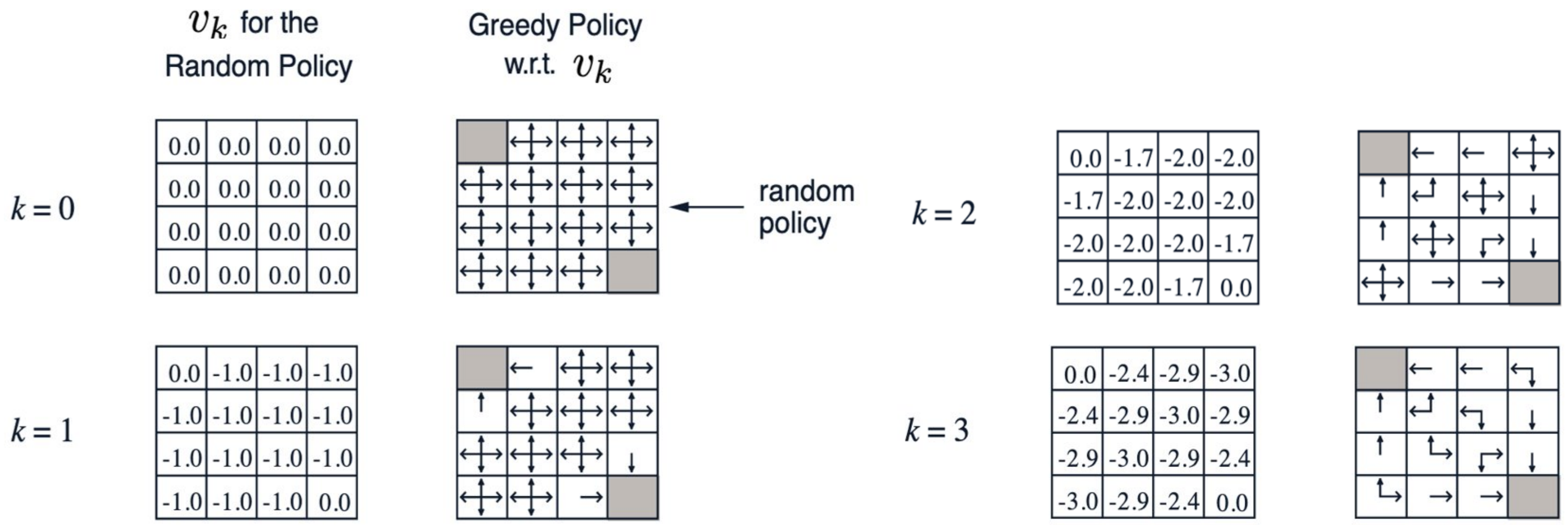
Find Optimal Policy: value based method

- There is always a deterministic optimal policy for any MDP
- If we know $Q^*(s, a)$, we immediately have the optimal policy
- How to find Q^*?
  - Policy Iteration / Value Iteration



Find Optimal Policy: value based method

Policy Iteration / Value Iteration



$$V^{\pi}(s) = \sum_{a} \pi(a|s)\Big(r(s,a) + \gamma \sum_{s'} P(s'|s,a) V^{\pi}(s')\Big)$$

$$\pi(s) = \arg\max_{a} Q(s,a)$$

Find Optimal Policy: value based method

Q-Learning

1. Initialize Q table, learning rate α
2. At state s, choose action $a = \arg\max_a Q(s,a)$ (may add noise for exploration)
3. Observe reward r and next state s'
4. $Q(s,a) \leftarrow Q(s,a) + \alpha\big[r + \gamma \max_{a'} Q(s',a') - Q(s,a)\big]$

Time difference (TD) error: $r + \gamma \max_{a'} Q(s',a') - Q(s,a)$

Find Optimal Policy: value based method

Deep Q-Learning

- When $\mathcal{S}$ and $\mathcal{A}$ spaces are large, it is impossible to maintain a Q table.
- Use function approximation to represent $Q(s, a)$, e.g., a deep neural network with parameter θ.
- Update θ to minimize the TD error.
    - Take derivative for θ and gradient descent

Find Optimal Policy: policy gradient method

- When action space is continuous, it is unable to $\arg\max_a Q(s,a)$
  - PG still works when action space is discrete
- Goal is to find optimal policy, can we directly parametrize the policy and optimize?

$$\pi_\theta(a|s) = \mathbb{P}[a|s,\theta]$$

- Start from some initial state, execute policy up to some time horizon T.

$$\tau = (s_0, a_0, r_1, s_1, a_1, r_2, \ldots, s_{T-1}, a_{T-1}, r_T, s_T) \quad R(\tau) = \sum_t \gamma^t r_{t+1}$$

- Maximize $\mathbb{E}_\tau[R(\tau)]$
- Need to take derivative for θ and do gradient ascent

Find Optimal Policy: policy gradient method

Policy Gradient

$$\nabla_\theta \mathbb{E}_\tau[R(\tau)] = \mathbb{E}_\tau[\nabla_\theta \log P(\tau|\theta) R(\tau)] \quad \text{apply} \quad \nabla_\theta \log(x) = \frac{1}{x}\nabla_\theta x$$

$$P(\tau|\theta) = \mu(s_0) \cdot \prod_{t=0}^{T-1} [\pi_\theta(a_t|s_t) \cdot P(s_{t+1}|s_t, a_t)]$$

$$\log P(\tau|\theta) = \log \mu(s_0) + \sum_{t=0}^{T-1} \log[\pi_\theta(a_t|s_t) + P(s_{t+1}|s_t, a_t)]$$

$$\nabla_\theta \log P(\tau|\theta) = \nabla_\theta \sum_{t=0}^{T-1} \log \pi_\theta(a_t|s_t)$$

$$\nabla_\theta \mathbb{E}_\tau[R(\tau)] = \mathbb{E}_\tau[R(\tau) \nabla_\theta \sum_{t=0}^{T-1} \log \pi_\theta(a_t|s_t)]$$

Find Optimal Policy: policy gradient method

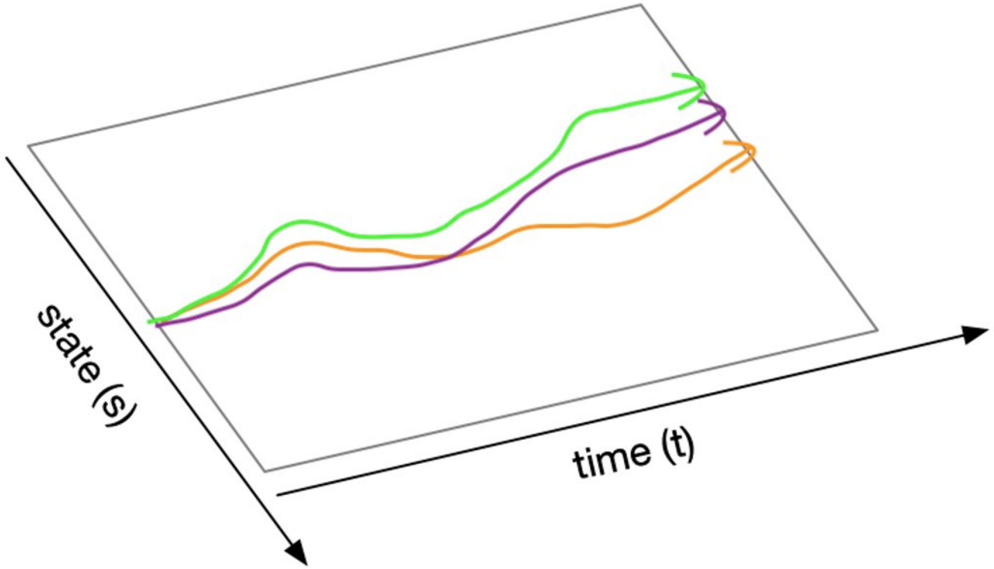
Vanila Policy Gradient

- Initialize a policy π_θ, learning rate α
- sample $\tau = (s_0, a_0, r_1, s_1, \dots)$ by executing π_θ
- $\theta \leftarrow \theta + \alpha \nabla_\theta \mathbb{E}_\tau [R(\tau)]$

This gradient has high variance

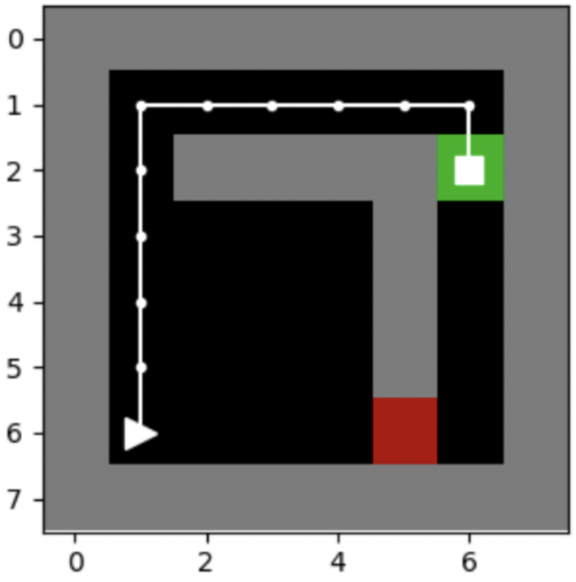
Risk-neutral vs Risk-averse RL

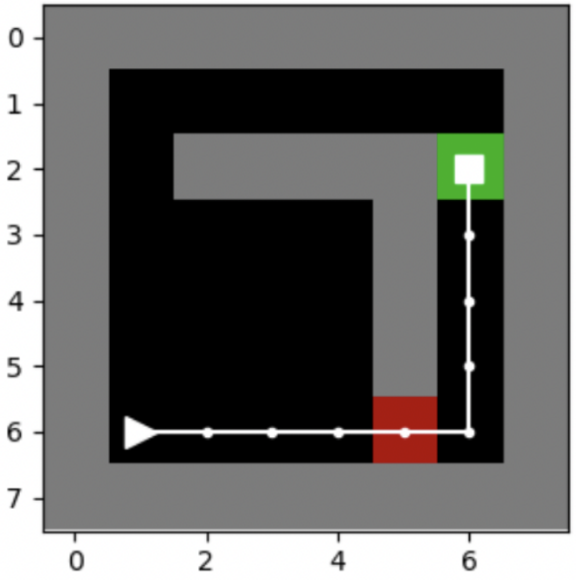
The cumulative return, i.e., $R_1 + \gamma R_2 + \gamma^2 R_3 \ldots$, is a random variable.



- Risk-neutral RL: only maximize the expectation (mean)
- Risk-averse RL: maximize mean while minimize some risk term, e.g. variance

Mean-Variance RL: example





- -1 per step, except
- red state: $\{-10, -1, 8\}$ with prob $\{0.4, 0.2, 0.4\}$.
    - $(-10) \times 0.4 + (-1) \times 0.2 + 8 \times 0.4 = -1$

Mean-Variance RL

$G = R_1 + \gamma R_2 + \gamma^2 R_3 \ldots$

$$\max_{\pi} \mathbb{E}[G] - \lambda \mathbb{V}[G]$$

$\mathbb{E}[G]$: time consistent, Bellman equation, TD learning

$\mathbb{V}[G]$: time inconsistent, minimizing variance at each step is not minimizing variance of the total return

Unable to use valued based method, consider policy gradient?

Mean-Variance RL Policy Gradient

(Switch among π, θ, π_θ)

$\mathbb{V}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

Define $J(\theta) = \mathbb{E}_\pi[G]$, $M(\theta) = \mathbb{E}_\pi\left[(\sum_{t=0}^{\infty} \gamma^t R_{t+1})^2\right]$

$$J_\lambda(\theta) = J(\theta) - \lambda\big(M(\theta) - J^2(\theta)\big)$$

$$\nabla_\theta J_\lambda(\theta) = \nabla_\theta J(\theta) - \lambda\big(\nabla_\theta M(\theta) - 2J(\theta)\nabla_\theta J(\theta)\big)$$

- $\nabla_\theta J(\theta)$: $\mathbb{E}_\tau[R(\tau)\omega(\theta)]$ $\qquad \omega(\theta) = \nabla_\theta \sum_{t=0}^{T-1} \log \pi_\theta(a_t|s_t)$
- $\nabla_\theta M(\theta)$: $\mathbb{E}_\tau[R^2(\tau)\omega(\theta)]$
- $J(\theta)\nabla_\theta J(\theta)$: need double sampling

Mean-Variance RL Policy Gradient

Tamar et al. (2012): two time scale algorithm

- A faster learning rate for estimating $\mathbb{E}[G]$ and $\mathbb{V}[G]$.
- A slower learning rate for updating θ.
  - $J(\theta)\nabla J(\theta)$: $J(\theta)$ uses $\mathbb{E}[G]$, only compute $\nabla_\theta J(\theta)$

Mean-Variance RL Policy Gradient

Xie et al. (2018): introduce Fenchel duality: $x^2 = \max_y(2xy - y^2)$.

Define $J(\theta) = \mathbb{E}_\pi[G]$, $M(\theta) = \mathbb{E}_\pi\big[(\sum_{t=0}^\infty \gamma^t R_{t+1})^2\big]$

$$\max_\theta J_\lambda(\theta) = J(\theta) - \lambda\big(M(\theta) - J^2(\theta)\big)$$

$$\begin{aligned} F_\lambda(\theta) &= (J(\theta) + \frac{1}{2\lambda})^2 - M(\theta) = \frac{J_\lambda(\theta)}{\lambda} + \frac{1}{4\lambda^2} \\ &= \max_y \left(2y(J(\theta) + \frac{1}{2\lambda}) - y^2\right) - M(\theta) \end{aligned}$$

No $J(\theta)\nabla_\theta J(\theta)$ term!

Upper bound of total return variance

Regard per-step reward as a random variable R (Bisi et al., 2020)

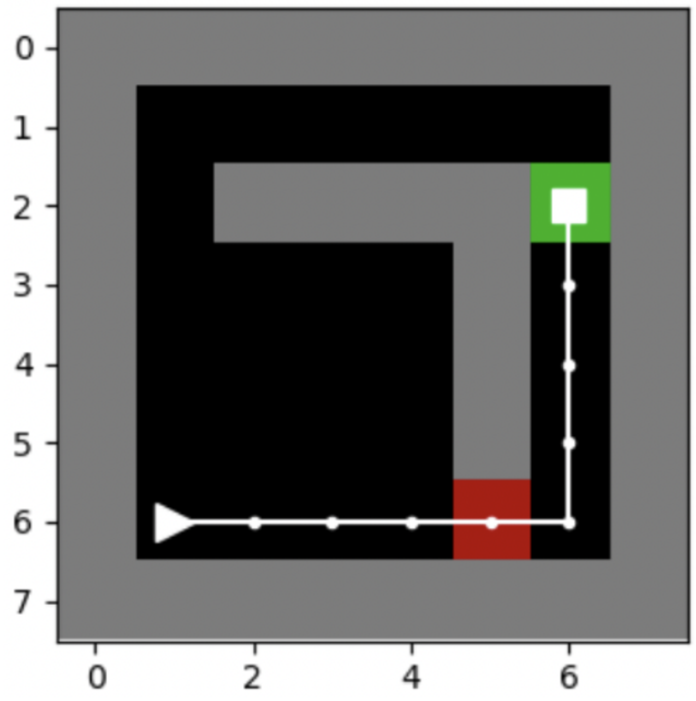
$$\mathbb{V}[G] \leq \frac{\mathbb{V}[R]}{(1-\gamma)^2}$$

$\gamma = 0.99$

-1 per step, 10 for goal, red state $\{-10, -1, 8\}$

$\{-1, \ldots, -10, \ldots, 10\}, p = 0.4, g \approx -7.23$
$\{-1, \ldots, -1, \ldots, 10\}, p = 0.2, g \approx 1.50$
$\{-1, \ldots, 8, \ldots, 10\}, p = 0.4, g \approx 10.23$

$\mathbb{V}[G] \approx 61.01, \mathbb{V}[R]/(1-0.99)^2 \approx 19.15/(0.01)^2$



Upper bound of total return variance

Benefit?

$$\hat{J}_\lambda(\pi) = \mathbb{E}[R] - \lambda\mathbb{V}[R] = \mathbb{E}[R] - \lambda(\mathbb{E}[R^2] - (\mathbb{E}[R])^2)$$

Fenchel duality (Zhang et al., 2021)

$$\hat{J}_\lambda(\pi) = \mathbb{E}[R] - \lambda\mathbb{V}[R] = \mathbb{E}[R] - \lambda\mathbb{E}[R^2] + \lambda \max_y(2\mathbb{E}[R]y - y^2)$$

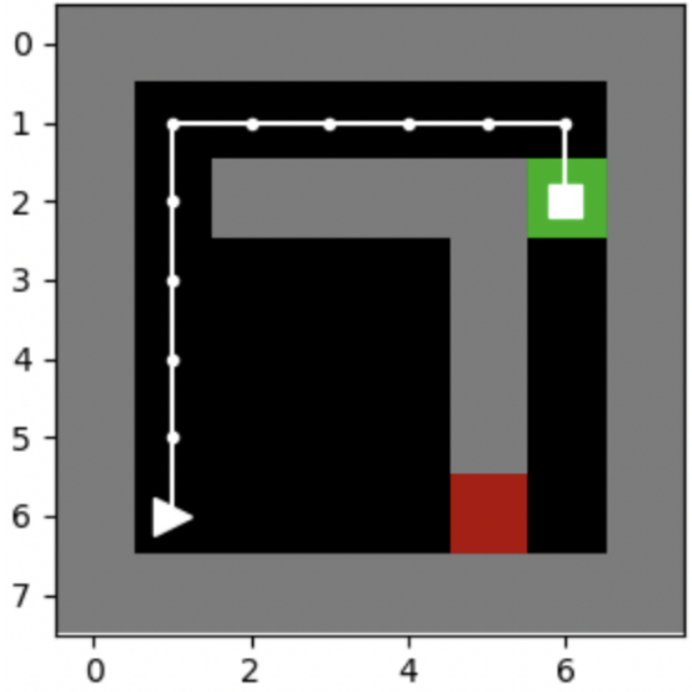
- close form solution for y (quadratic): $y^* = \mathbb{E}_{s,a\sim d^\pi}[R(s,a)]$
- New reward $\hat{r}(s,a) = r(s,a) - \lambda r(s,a)^2 + 2\lambda r(s,a)y$ to optimize for π.

Why? (occupancy measure)

$$\mathbb{E}_\pi[G] = \mathbb{E}_{s,a\sim d^\pi(s,a)}[R(s,a)]$$

Limitations

- Directly optimize mean-variance: double sampling issue
- Use $\mathbb{V}[R]$:
  - Different term, e.g., $\mathbb{V}[R] \neq 0$ while $\mathbb{V}[G] = 0$
  - Sensitive to reward magnitude



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